Abstract

According to modern theories of stellar evolution, white dwarves are the last stage of evolution for all stars with less than 4 times the mass of the sun. These stars are in an equilibrium state, between the force of gravity pulling inward, and the pressure from degenerate electrons pushing outward. In 1931 Chandrasekhar showed that as a white dwarf became more massive, and the electrons that supported its weight became relativistic, there would be a point beyond which the degeneracy pressure would be insufficient to support the star. This mass is approximately 1.4 times the mass of the sun, it is known as the Chandrasekhar limit.

In this paper I will discuss the history of the discovery, and its importance in astrophysics, then derive in detail the Chandrasekhar limit, including some of the refinements that have been made since Chandrasekhar's original paper.

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Section I - History

This section will discuss the events leading up to the discovery of the Chandrasekhar limit. This will begin with a brief historical account of Chandrasekhar's life, in order to appreciate how his work influenced modern science.

Subrahmanyan Chandrasekhar, known to many as Chandra, was born in Lahore on October 19th, 1910. He was the eldest son of C.S. Ayyar, his father, and Sitalakshmi, his mother. When Chandra was six, the family moved to Lucknow, in northern India. Two years later, when his father became deputy accountant-general, they moved again, this time to Madras.

Chandra's parents began his education at home, at age five. This was common among middle and upper class families, since the schools were very poor. He did not attend formal school until 1921, when he was eleven. He was accepted into the third year of high school, skipping two full years. The Hindu High School he attended was considered the best school in Madras. Chandra did extremely well in high school, and became a freshman at the Presidency College in Madras at only fifteen years of age. He was considered a prodigy, especially in mathematics. His private studies in mathematics put him far ahead of his classmates, and he invariably received the highest grade in the class. In college he studied physics, chemistry, English, and Sanskrit. He found himself drawn most to physics, and English.

After completing his intermediate two years with distinction in physics, chemistry, and mathematics, his next step was to work toward a B.A. honors degree. Chandra's first choice was mathematics. However his father insisted on physics, seeing no future in mathematics. When school started, he became a physics honors student, but he attended lectures in the mathematics department. He studied the prescribed physics texts on his own, and took all the required tests.

In 1928, after completing the first year of the three year honors program, Chandra went to work in his uncle's laboratory. His uncle, Raman, known for discovering the Raman effect in the molecular scattering of light, wanted to give his nephew a start in experimental physics. Chandra was to assist in an experiment on X-ray diffraction by liquids. After only a week, however, Chandra had broken the apparatus, and it was decided that his future did not lie in experimental physics.

During the beginning of his second year of honors studies Arnold Sommerfeld came to Presidency college to give a lecture. Chandra spoke to Sommerfeld after the lecture, and was informed that much of the physics he had learned had been changed by new developments due to Heisenberg, Dirac, Pauli, and others. Sommerfeld left Chandra with some unpublished material of his, on the theory of electrons in metals. Chandra immediately launched into a study of the new material. In a few months Chandra had written a paper of his own, The Compton Scattering and the New Statistics. He felt that the paper was good enough to publish, so he contacted Ralph Howard Fowler, a Fellow with the Royal Society. Fowler had applied the new Fermi-Dirac statistics to an entirely different area - astrophysics, specifically to white dwarf stars. In the future Fowler's paper was to have a great impact on Chandra's life and career. In 1929 Chandra heard back from Fowler, and after a few minor changes, the paper was published in the Proceedings of the Royal Society later that year.

During his final undergrad year, Werner Heisenberg came to Madras. Chandra was in charge of the visit, and spent the day showing him around. This gave Chandra the chance to discuss his ideas, and led to Chandra meeting many important people in his field.

Chandra received a full scholarship to Cambridge, that had been created specifically for him. The only stipulation was that Chandra had to agree to return to India. Chandra considered not going, because of his mother's ill-health, but she encouraged him to go. He left from Bombay on July 31st 1930. Just before he departed, he had completed a paper. In it he had developed further Fowler's theory of white dwarves.

At this point it is appropriate to pause, and take a look at the developments in the theory of white dwarves that had taken place prior to Chandra's involvement. Between 1834 and 1844, the astronomer and mathematician F. W. Bessel found that the star Sirius had wavy irregularities in its motion through space.[1] He concluded that it had an invisible companion revolving about it with a period of about 50 years. It failed to show itself, however, until January 1862, when the discovery was made by Alvan G. Clark, using an 18 1/2-inch refracting telescope, then the largest refractor in the world.[2] The companion, now called Sirius B, was found to be magnitude 8.65. This meant that although it had a mass comparable to the sun, its luminosity was less than 1/400th that of the sun. The abnormally low luminosity might be explained in two ways, either by an extremely low surface temperature, which would imply a low surface brightness, or by an unusually small diameter. The spectrum of the star was difficult to determine due to the proximity of Sirius. In the absence of other evidence, it was generally supposed that the star must be cool, since a diameter small enough to explain the observed luminosity would imply a density 125,000 times greater than water. This was, at the time, considered impossible.

In 1915 the mystery deepened. Walter S. Adams announced, that as the companion passed to the furthest distance from the primary in its 49 year orbit, he had succeeded in securing a spectrogram with the Cassegrain reflector at Mt. Wilson.[3] It showed a spectrum identical to Sirius. Since stars were known to radiate, to a fair approximation, like black bodies, with temperatures well correlated with color, the dimness of Sirius B could not be explained as low surface brightness. Although some suggested that it might be reflected light from Sirius, Adams pointed out that a similar star had been found with no companion. Some months earlier, the Danish astronomer Ejnar Hertzsprung had discovered that Omicron-2 Eridani was under-luminous in much the same way.

In 1916 the Estonian astrophysicist, Ernest J. Opik calculated the mean density for 40 stars.[4] He noted that Omicron-2 Eridani seemed to have a density of 25,000 times that of water, but also noted that this was "impossible".

Finally in 1924 Eddington speculated that such densities might be possible for ionized material, since the electrons that marked the boundary of the atom were not present.[5] He contacted Waler Adams at Mt. Wilson, and asked him to measure the radial velocity of Sirius B, with the aim of determining the density by measuring the relativistic shift in the spectral lines. Fifteen months later, in 1925, Adams published his conclusions, which supported a very high density for Sirius-B.

However, this presented another problem. Since the star could only exist this way in an ionized state, and since it was thought that as it cooled it must return to the form of ordinary matter, the star would need to expand as it cooled. The star would need energy to cool.

An answer came in 1926, when Fowler pointed out that using the newly discovered quantum statistics, even an absolutely cold assembly of electrons, confined to a finite volume would have a finite pressure.[6] Then, in 1930, Milne showed that this zero-point pressure can balance a cold star against gravity, at a uniquely determined radius that corresponds well with the actual sizes of white dwarves.

It is at this point that Chandra came into the picture. He had combined Fowler's ideas with Eddington's work on stellar bodies in equilibrium between gravity and their own internal pressure, and had obtained a more detailed picture of a white dwarf star. He concluded that the central density of such a star would be about six times its average density. Then during the long trip from India to England it occurred to him that at such high densities relativistic effects might be important. He quickly found that this was indeed the case. Chandra began working, expecting to find a relativistic generalization of Fowler's theory. But, to his surprise he found something totally different. He found that there was a limit to the mass of a star that would evolve into a white dwarf. This limit involved only fundamental constants and the average molecular weight of the stellar material. He resolved to write a paper on the results, and discuss it with Fowler at the earliest opportunity. He knew that a great deal of work would be necessary to understand the result fully, and to establish it on firm ground.

When Chandra got to London there was a great deal of confusion about his admission. Fowler, with whom he had had prior communications, was away in Ireland. He was finally admitted 2 1/2 weeks later on the strength of a personal recommendation from Fowler. He later realized how fortunate it was for him that he had written to Fowler, two years before. Without him, he probably never would have been admitted.

On October 2nd, Chandra met with Fowler to present him with the two papers he had written. The first paper, which was an extension of Fowler's work, impressed him very much, however, he was unsure of Chandra's paper that included relativistic effects, and said that he would send the paper to Edward Arthur Milne. Fowler also advised Chandra as to which classes would be most valuable to him, including a class in quantum mechanics taught by Paul Dirac. He did attend this class, and also a class on Relativity, taught by Eddington.

Chandra came to know Dirac quite well. Dirac became Chandra's advisor when Fowler took a sabbatical at the end of Chandra's second term. Dirac often came to Chandra's room for tea on Sunday's, and they got in the habit of taking long walks together on the old Roman roads. Dirac did not have much interest in astrophysics, and so, was not of much help to Chandra in his work. Chandra continued to do research on his own however, and submit papers for publication. His work won him election to the "Sheep Shanks Exhibition", a special award given each year to one candidate in astrophysics.

About this time Chandra began communications with Milne. Although he had received no response to his paper on the critical mass of white dwarves, Milne was quite receptive to his subsequent work on stellar atmospheres and relativistic ionization. He soon developed a strong rapport with Milne, and collaboration and joint publication was suggested.

It was also during his first year that Chandra was introduced to the meetings of the Royal Astronomical Society (RAS) by Fowler. These meetings were an important part of Chandra's career at Cambridge. He was asked several times to present his work to the society.

Chandra made few friends at Cambridge, being much too busy most of the time for socializing. Two exceptions to this rule however, were Chowla, and Harold Grey. Chowla was another student from India, who shared Chandra's work ethic. Harold Grey was a physicist who was involved in a pacifist movement that was sympathetic to Gandhi's struggle for Indian independence. Grey provided Chandra with a link to the world of pure physics.

Chandra was also very strict about his diet in England. He insisted on remaining vegetarian, although he did start eating eggs. He found the non-meat diet in England extremely bland and very limited.

On May 21st, 1931, Chandra received news from his father, that his mother had passed away. Chandra was very upset by the news, and being so far from home made things worse for him. On May 28th he wrote to his father: "Time helps to heal wounds ever so sore they may be. That appears to me the tyrannous aspect of time. However I have consoled myself sufficiently to begin the daily work."[7] In fact Chandra had had his first meeting with Eddington the day after he had received the news.

In order to help himself recover he thought a change of scenery would help. So, he choose to go to Gottingen, Germany, where he studied at the Institut fur Theoretische Physik, with Max Born as its director. There, among other things he studied quantum mechanics with Born, and wrote two papers that summer.

He also visited Potsdam Observatory, in the suburbs of Berlin, and met Erwin Finlay Freundlich, who was at the time a well known astrophysicist. He was surprised to find that Freundlich recognized him, and knew of his work. In fact Freudlich invited him to be a guest lecturer.

In September Chandra returned to Cambridge. He immediately sent the paper on stellar coefficients of absorption that he had written in Germany, to Milne and the Royal Astronomical Society.

On November 12th, 1931, Milne contacted him about his paper. It was very much a continuation of Milne's own work, and he suggested that they work together on another paper. In December he presented this paper to the RAS, and received high compliments.

Chandra had ended up doing astrophysics by accident, and was becoming disheartened with the subject. Although Chandra felt that he had discovered something important in his work on the limiting mass of degenerate stars, he was not receiving positive feedback. He had always wanted to study pure math, however he felt that was impractical. Thus, he decided to focus his efforts on pure physics. He read some papers by Pauli and Heisenberg, and talked with Dirac, who advised him to go to Copenhagen to study with Niels Bohr. Chandra did just that in August, 1932.

Chandra enjoyed Copenhagen, and worked on a problem in physics that had been suggested by Dirac. In October he wrote a paper entitled "On the Statistics of Similar Particles", and sent it to Dirac. Both Bohr and Rosenfeld approved the paper, and it was sent to the RAS for publication. However, that winter Dirac discovered an error, and the paper had to be withdrawn.

About this time Milne was visiting from Cambridge. They discussed Milne's work on the equilibrium of rotating gas spheres, and Chandra was soon able to write a paper on the subject, which Milne agreed to report to the RAS. Also, about that time Chandra received an invitation to lecture at the University of Liege, Belgium, on various topics in astrophysics. In February he gave his six lectures, he was honored with a bronze medal, and urged to publish his lectures in the academy of Sciences proceedings. This returned Chandra fully to the world of Astrophysics.

In March Chandra returned to Cambridge, because his thesis was due in June. He spent his time working on a series of papers on distorted polytropes, which was more than adequate material for his thesis.

Chandra was not totally happy about the results of his trip to Denmark. He went there to make a change, but was forced to return to astrophysics. However, he realized that some things had to be done in order to further his career. He was becoming very concerned about what his future held. The government scholarship was almost over, and according to the agreement, he must return to India. He wanted to find a place in India where he could continue the research he wanted. However, such positions did not exist unless they were specially created, and the Director of Public Instruction, who had promised to create a position for him, was no longer in charge.

He wrote to the Director to seek an extension of the scholarship, but had no success. He became determined to extend his stay in Europe.

Chandra turned his thesis in to Fowler on May 17, 1933. Fowler did not even glance at it, but told him to bring it to the Registrar, saying he had full confidence in Chandra's work.

The oral examination was a mere formality, as he had already presented a summary of his work to the RAS. Fowler and Eddington were his examiners, they did manage to worry him somewhat, but Chandra passed easily.

It was suggested that Chandra might try to apply for a fellowship at Trinity College. However, since it was open to candidates from all fields, the competition was quite severe. Never-the-less, it would solve his problem, so he applied.

On October 9, 1933 the notice was posted listing the fellowship recipients. Chandra's name was on the list, much to his surprise. Later, Milne disclosed to him, that he had been called in as a referee. Although, he made it clear that Chandra's work had got him the position, not Milne's influence.

Chandra was elated, this meant at least three more years in Europe. The fellowship was actually for four years, but the last year could be spent anywhere. Chandra's father was not entirely pleased with his decision to stay in Europe, but he adjusted to the idea. Then, he decided to visit Europe himself. Chandra was not totally pleased with the idea, and tried to discourage him, but his father was not easily discouraged.

His father visited and toured Europe for six months, and although the visit went well, Chandra was glad when it was over.

For the time being Chandra stopped thinking about changing fields. He was formally admitted to the RAS. He had previously only attended as the guest of other members.

The seating was hierarchical. with senior members like Eddington and Jeans in front. Chandra got to sit in the last row. He often wondered what would happen if he should sit in front.

In 1934, Chandra got the opportunity to visit Russia. He traveled by boat to Leningrad, and he clearly remembers the numbers of German warships seen on the way. In Leningrad he gave two lectures at the Pulkovo Observatory, to large audiences. One lecture was on white dwarves. A friend he had made there, Viktor Ambartsumian encouraged him to look into the matter in more detail, by eliminating some approximations. Chandra was encouraged, and did just that.

A year later Chandra was distressed to hear that all the Russian scientists had been sent to Siberia, or killed. Ambartsumian, it turns out had been lucky and escaped, and Chandra got to meet him again in 1981, at a symposium in Russia.

He returned full of determination. He spent the next few months doing detailed calculations, in order to obtain an exact theory. He proved beyond any reasonable doubt, that the limit on the mass of a star that could become a white dwarf was unavoidable.

Chandra also hoped that his work would help to resolve a long standing conflict between Eddington and Milne. Eddington's standard model considered the surface of the star to be of little importance, compared to the center. And the model assumed a zero temperature at the star's surface, for simplicity. Milne argued that the surface may be important, and that the star may not be a perfect gas throughout. Since Eddington would have nothing to do with these ideas, Milne went on to form more extreme models which required every star to have a degenerate core. With this in mind, Milne was unwilling to judge Chandra's work on its own merit, looking instead for verification of his own ideas.

Since his work was being ignored in England, Chandra published in the Astrophysical Journal of America. Chandra knew that his results showed all of Milne's work to be wrong, so he kept quite about it, in general.

In 1932, Chandra was visiting Copenhagen. There Leon Rosenfeld urged Chandra to publish his work. Knowing of Milne's objections, Chandra decided to send his article to Potsdam to be published in the Zeitschift fur Astrophysik. Unfortunately, Milne was visiting Potsdam at the time, and was asked to review the paper. He did not recommend its publication until Chandra had written him a long letter of explanation.

For the next two years he worked on other problems, since there seemed to be little interest in his result. In 1934, however he felt ready to tackle the problem again, this time removing as many assumptions as he possibly could. Eddington took much interest in his work at this point. Chandra assumed at the time, that this was because his results would prove Eddington correct, and Milne wrong. As it turned out Eddington thought that the precise formula would eliminate the collapse.

At the end of 1934, Chandra submitted two papers on his results to the RAS, and he was scheduled to deliver his papers at the next meeting. He discovered a couple of days in advance that Eddington would be giving a paper on "Relativistic Degeneracy" immediately after his. This was somewhat annoying, since Eddington had said nothing about it.

After Chandra gave his paper, Eddington said that he believed that what Chandra had proved was that the relativistic degeneracy formula must be wrong, since stars should not behave in this absurd way. And he went on to give his ideas on how the formula might be wrong.

Chandra felt humiliated. Months of work had been swept aside as being fundamentally in error. And, of course, because of Eddington's authority, most believed that Chandra must have been wrong.

Chandra decided to try to get a definitive statement from a leader in quantum physics, that the formula he had used was the correct one. He wrote to Rosenfeld in

Copenhagen, and asked him to speak to Bohr. Rosenfeld wrote back, that Bohr had looked at it, and that they had both agreed, that the formula used was definitely correct.

Armed with this Chandra tried various arguments to try to convince Eddington, but nothing worked, and some people urged him to just drop the matter.

Chandra wrote again to Rosenfeld, and asked if Bohr could read his paper, and make a definitive statement. Bohr responded that he was currently too busy, but that he should send the paper to Pauli. Chandra did this, and Pauli agreed that there was nothing wrong with the formula, but he was also unwilling to get involved in the conflict. Writing to Dirac produced similar results.

Meanwhile Eddington was continuing to attack his work. In 1935 Eddington gave an hour long talk to the International Astronomical Union, and devoted most of it to saying that Chandra was wrong. Chandra asked to respond, but his request was denied.

Eddington's arguments tended to shift back and forth, as various points were made, but he continued to believe that there was no limiting mass.

Chandra didn't know what to do. He had planned on working out theories for rotation, and other things, but felt a lack of enthusiasm now, since his results would not be taken seriously.

In 1939, both were invited to speak at an international meeting in Paris, on white dwarves and novae. Chandra gave his talk first, followed by Eddington the next day. At one point Chandra became very angry over a comment Eddington made. Eddington later apologized, but Chandra was still rather unhappy about it.

As it turned out, this was their last meeting. Eddington died in 1944. Chandra later said that he regretted not being more forgiving the last time they met. In spite of the professional disagreement, they had remained friends.

Some think that the theories of white dwarves, neutron stars, and black holes, could have been as much as 20 years further along, if Eddington had excepted Chandra's ideas, but it is impossible to tell what really would have happened.

Chandra moved on, and pursued other things, and eventually the scientific community realized he was right. In 1983 he was given the Nobel prize for his work.

Section II - Qualitative Discussion

Modern theories suggest that white dwarf stars are the final state for most stars. A star spends most of its existence in a state of equilibrium, between the gravitational force, trying to pull it together, and the thermal pressure from the nuclear reactions at its core, trying to push it apart. When its nuclear fuel is exhausted, it must contract. The end product of this collapse, for low mass stars is a white dwarf. White dwarves themselves can be no more massive than 1.4 times the mass of the sun. However, it is believed that the progenitor star can be as much as 4 times as massive as the sun, because during the end of their nuclear fuel burning stage most stars eject a large portion of their mass. In the case of low mass stars, this leads to a planetary nebula. A small minority of stars, that are more massive than 4 times the mass of the sun, will probably not end up as a white dwarf. These stars collapse even further, becoming neutron stars, or black holes. Their final moments as "normal" stars are more spectacular as well. Most of them, it is believed, will finish their lives in a supernova explosion. Some of our basic knowledge about white dwarves is summarized in this section.

White dwarves have very small diameters, closer to the size of planets, than that of normal stars. Sirius B has a diameter of 19,000 miles, compared to the earth's diameter of 7900 miles. The smallest white dwarves yet observed are believed to have diameters in the range of 1000 miles.

White dwarves are under-luminous compared to normal stars. Sirius B has an absolute magnitude of 11.4 or 435 times fainter then our sun's magnitude of 4.8. White dwarves range in magnitude from about 9.0 to 16.0. The brightest known is approximately 8.9 or only about 1/40th of the sun's brightness. The dimmest may be fainter than 17th magnitude, or about 100,000 times fainter than the sun.

White dwarves have a wide range of temperatures, from 70,000 K to less than 5000 K, although most fall between 8000 and 10,000 K, giving them a white spectrum of class A. This is the origin of the name "white dwarf". The coolest white dwarf yet discovered is type K. No type M stars have been found. This has implications for the age of the universe, since the coolest white dwarves are believed to be the remains of the first stars formed. There may be no class M white dwarves, because no stars have yet had time to cool that much.

The masses of white dwarves have only been determined accurately in three binary systems. These are: Sirius B, 40 Eridani B, and Procyon B. They have masses of 0.98, 0.44, 0.65 solar masses respectively. The masses of other white dwarves have been estimated by theoretical models. The most massive are about 1.2 solar masses. The least massive are about 0.2 solar masses.

The densities of white dwarves are, of course, very high. Sirius B has a density of about 125,000 g/cm^3. The densest may be as much as 10,000 times denser than this. The most dense materials on earth are only about 20 g/cm^3. This is why the idea was initially regarded with skepticism. These densities would be unexplainable, without knowledge of quantum mechanics, and the structure of the atom.

White dwarves are sometimes found with planetary nebulae around them. And, almost all planetary nebulae are observed to have white dwarves at their center. Planetary nebulae got their name from the fact that when they were found, they looked similar to the planet Uranus through a telescope. A planetary nebula is the remains of the outer layers of the star that formed the white dwarf. The current theory for their formation says that when the outer layers of a red giant predecessor become cool enough for ionized atoms to recombine, the star's outer layers become unstable. The instability occurs because the temperature for the reaction is borderline, and the recombination releases energy. So, the outer layers contract, and expand, becoming warmer and cooler, and eventually the layers are ejected into space, forming a nebula. The nebula is visible because it is heated by ionizing radiation, from the hot central star. They tend to appear as rings around the stars, but are really semi-transparent shells, that totally surround the stars. Most nebulae are less than 50,000 years old, which fits in well with theories of their formation.

Spectroscopic studies of white dwarves are difficult to interpret. The stars are surrounded by a non-degenerate layer, which may be 50 to 60 miles thick, and above that is a layer of atmosphere, only about 100 feet thick. This is the only area observable spectroscopically. Also making things more difficult, the huge surface gravity, often more than 50,000 times that of earth, produces spectral peculiarities, such as a widening of the lines, often 20 to 50 Angstroms at half the central line depth.

The most common type observed is type DA. About 2/3 of white dwarves fall into this category. Their spectra show only one type of line, the hydrogen Balmer series of lines. These stars have been explained as having "settled". That is, the heaver elements have sunk lower in the star, leaving only hydrogen at the surface.

The formation of other types of white dwarves has not been satisfactorily explained. Type DB white dwarves, about 8% of all white dwarves, show only helium lines. They are also among the hottest of all white dwarves. This seems to indicate that their predecessor stars ejected all of their hydrogen, although no correlation between type DB stars, and planetary nebulae has been found. Also, these stars may be related to other types of hydrogen deficient stars that have been observed.

Type DC stars, about 14% of the total, show no lines at all. It has been shown that if a star's outer layer were pure helium, as it cooled the helium lines would disappear. Thus, DC stars may be cooled type DB stars. Also extreme widening of the bands has been mentioned as a possible way of explaining the absence of all lines.

The other categories of white dwarves are rarer, together only about 12% fall into these categories. Type DF stars have hydrogen and calcium lines at temperatures above 8000K, but below 8000K only a few lines of calcium and magnesium show. These stars may have been more evolved than other stars, and had heavier elements in their cores. Type DG stars are similar to type DF stars, but show lines of iron and calcium. These stars tend to be some of the coolest white dwarves. Some other stars show a band at 4670 Angstroms, this has been attributed to the carbon molecule, C_2. Others show an unidentified line at 4135 Angstroms. Also, two stars show emission lines, which are of interest, since one of them, WZ Sagittae, is known to be a recurrent nova.

Novae are associated with white dwarf stars in binary systems. Material from the normal companion star falls onto the surface of the white dwarf. After enough is built up, a run-away nuclear reaction takes place, briefly brightening the star by several orders of magnitude.

Some other interesting, resent discoveries about white dwarves include: A discovery that at the pressures and temperatures found in the cores of cooler white dwarves, carbon will form a crystal lattice of ions, with the degenerate electrons moving freely about. These stars would be, almost literally "diamonds in the sky"[8]. The light from some white dwarves is circularly polarized, this is an indicator of a strong magnetic field. The field strengths seem to be in the range of 10^6 to 10^8 gauss. A type of white dwarf known as a ZZ Ceti star, has been observed to oscillate. The periods of these oscillations are on the order of a few hundred to a few thousand seconds. ZZ Ceti stars have temperatures of 10,000K to 14.000K, and it is interesting to note that this occurs in the same instability zone as Cephid variables, when extrapolated into the region of white dwarves.

Only a few hundred white dwarves are known because they are very faint objects, but the numbers that are observed fit in well with theories of stellar death rates. The search continues for more of these objects, and the known white dwarves are intensely studied. They continue to be among the most interesting objects in astrophysics.

The first step in deriving the limit is to arrive at a relation between pressure and density for fermions, beginning with the Fermi-Dirac function.

(3.1)
$$f(E) = \frac{1}{\exp((E - \mu)/kT) + 1}$$

This expression gives the probability of a quantum state being occupied, for ideal fermions. Here k is the Boltzmann constant, T is the temperature, E is the total energy of the quantum state, and mu is the chemical potential, defined by,

(3.2)
$$\mu = \left(\frac{\partial \varepsilon}{\partial n}\right)_{S,V}$$

Where n is the number density, epsilon is the total energy density, V is the volume, and S is the entropy. It is important to note here that both E and epsilon refer to the total energy, including the rest mass, defined by,

(3.3)
$$E = (p^2 c^2 + m^2 c^4)^{1/2}$$

Here c is the speed of light and p is the momentum of the particle. This is important because relativistic velocities will be discussed in the derivation.

At low densities and high temperature Eq. (3.1) reduces to the Maxwell-Boltzmann distribution,

(3.4)
$$f(E) \approx \exp\left(\frac{\mu - E}{kT}\right)$$

However in the low temperature case, as T approaches zero, mu is called the Fermi energy E_F, and the probability of a state being occupied is given by,

(3.5)
$$f(E) \approx (1, E \leq E_F) f(E) \approx (0, E > E_F)$$

Now, the number density of quantum sites in a very small region of phase space can be given by,

(3.6) $$\frac{d\Pi}{d^3x\,d^3p}$$

Where pi is the total number in phase space.

Since h^3 is the volume of a cell in phase space, where h is Plank's constant, the probability of a cell being occupied can be expressed as,

(3.7) $$f(p) = \frac{d\Pi}{d^3x\,d^3p}\left(\frac{h^3}{g}\right)$$

Where g is the multiplicity factor that gives the number of quantum states corresponding to a given momentum. For electrons g = 2s+1= 2 where s is the spin (1/2).

Eq.(3.7) is equal to Eq. (3.1) since both are the probability of a state being occupied. The distribution function can be described either as a function of E, or as a function of p, since E and p are related by Eq.(3.3).

The number density in real space can be found by integrating over all momenta.

(3.8) $$n = \int \frac{d\Pi}{d^3x\,d^3p} d^3p$$

Or, using eq(3.7), equivalently by,

(3.9) $$n = \int \frac{g}{h^3} f(p)\,d\bar{p}$$

where dp is the volume element in momentum space.

The energy density of the gas can be found by taking the energy of a particle in a specific state, multiplying by the probability of the state being occupied, and integrating over all phase space.

(3.10) $$\varepsilon = \int E \frac{d\Pi}{d^3x\, d^3p} d^3p$$

Or alternately by,

(3.11) $$\varepsilon = \int E f(p) \frac{g}{h^3} d\bar{p}$$

Now, since pressure is a momentum flux, it can be written,

(3.12) $$P = \frac{1}{3} n < \bar{v} \cdot \bar{p} >$$

The < >'s indicate an average over all particles, and the factor of 1/3 arises from isotropy, that is $<v_x p_x> = <v_y p_y> = <v_z p_z>$. Here v is the velocity, given by $v = pc^2/E$. Also eq.(3.12) can be derived using two basic equations from thermodynamics, P = nkT, where again n is the number density, and $3kT/2 = mv^2/2$, where m is the particle's mass.

Now using eq.(3.8), the pressure can be written as,

(3.13) $$P = \frac{1}{3} \int pv \frac{d\Pi}{d^3x\, d^3p} d^3p$$

It can also be written as,

$$P = \frac{g}{3h^3}\int \bar{p}\cdot\bar{v} f(p) d\bar{p}$$

(3.14)

Now if it is assumed for simplicity that T = 0 , and that the particles involved are ideal fermions, these equations can be integrated. From eqs.(3.5) and (3.9), and using g = 2 for electrons, the result is,

(3.15)
$$n_e = \frac{g}{h^3}\int_0^{p_F} d\bar{p} = \frac{4\pi g}{h^3}\int_0^{p_F} p^2\,dp = \frac{4\pi g}{3h^3} p_F^3 = \frac{8\pi}{3h^3} p_F^3$$

Here p_F is the Fermi momentum, defined by,

(3.16)
$$E_F = (p_F^2 c^2 + m^2 c^4)^{1/2}$$

If m_B is defined as the mean baryon rest mass, Y_e as the mean number of electrons per baryon, and n_e as the electron number density, and assuming the mass is due almost entirely to baryons, the rest mass density is defined as,

(3.17)
$$\rho_0 = \frac{n_e m_B}{Y_e}$$

Note that rho_0 is not a costant. Now, substituting eq.(3.15) into eq.(3.17) gives,

(3.18)
$$\rho_0 = \frac{8\pi}{3h^3}\frac{m_B}{Y_e} p_F^3$$

Now the pressure can be found using eqs.(3.5) and (3.14),

(3.19)
$$P = \frac{2}{3h^3}\int_0^{p_F} \bar{p}\cdot\bar{v} d\bar{p} = \frac{2}{3h^3}\int_0^{p_F} 4\pi\, p^3 v dp$$

In the non-relativistic case $p=m_e v$ and the pressure can be given by, (3.20) $$P = \frac{2}{3h^3 m_e}\int_0^{p_F} 4\pi\, p^4\, dp = \frac{8\pi}{15 h^3 m_e} p_F^5$$

In the ultra-relativistic case v=c and the pressure can be given as,

(3.21) $$P = \frac{2c}{3h^3}\int_0^{p_F} 4\pi\, p^3\, dp = \frac{2\pi c}{3h^3} p_F^4$$

Now using eq(3.18) with eqs.(3.20) and (3.21) the relation between density and pressure can be expressed. In order to emphasize this relationship, it is put into polytropic form. This form is explained in detail in Appendix A. In this form, the pressure is given as,

(3.22) $$P = K\, \rho_0^{\gamma}$$

In the non-relativistic case,

(3.23a) $$K = h^2 \frac{Y_e^{5/3}}{m_B^{5/3}} (\frac{3}{8\pi})^{2/3} \frac{1}{5\, m_e}$$

and,

(3.23b) $$\gamma = \frac{5}{3}$$

And, in the ultra relativistic case,

(3.24a) $$K = h(\frac{3}{8\pi})^{1/3} (\frac{Y_e}{m_B})^{4/3} \frac{c}{4}$$

a

and,

(3.24b) $$\gamma = \frac{4}{3}$$

The next step after arriving at the pressure-density relation, is to find the relation between density and the radial distance from the center of the star. To start, consider a spherical shell inside a star, with radius r, thickness dr, density rho_0, and mass interior to the shell m(r). The gravitational force on this shell is,

(3.25) $$\frac{m(r)G}{r^2} 4\pi r^2 \rho_0 dr$$

Now, let dP be the pressure difference across dr, then the force due to the pressure supporting the shell is,

(3.26) $$-4\pi r^2 dP$$

Using the negative sign to indicate that it apposes the gravitational force.

Equating eq.(3.25) and eq.(3.26) gives,

(3.27) $$\frac{dP}{dr} = -\frac{m(r)G}{r^2} \rho_0$$

This is the equation of hydrostatic equilibrium.

Now, m(r) can be given by,

(3.28) $$m(r) = \int_0^r \rho_0 4\pi r^2 dr$$

or,

(3.29) $$\frac{dm(r)}{dr} = 4\pi r^2 \rho_0$$

Eq.(3.27) can be re-written as,

(3.30) $$m(r) = -\frac{r^2}{\rho_0 G}\frac{dP}{dr}$$

Now substituting eq.(3.30) into eq.(3.29) gives,

(3.31) $$\frac{d}{dr}\left(\frac{-r^2}{\rho_0 G}\frac{dP}{dr}\right) = 4\pi r^2 \rho_0$$

or,

(3.32) $$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho_0}\frac{dP}{dr}\right) = -4\pi G \rho_0$$

This is a density - radius relation, that involves pressure.

The next step is to take the pressure - density relation, eq.(3.22), and substitute it in to eq.(3.32), in order to get a density - radius relationship. The result is,

(3.33) $$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2 \gamma}{\rho_0} K \rho_0^{(\gamma-1)}\frac{d\rho_0}{dr}\right) = -4\pi G \rho_0$$

This is a second degree differential equation for the density. It can be solved with the boundary conditions: rho_0 = rho_c, the central density, at r=0, and rho_0 = 0 at r=R, the surface of the star. The next step will be to put the equation into dimensionless form. First let,

(3.34)
$$\gamma - 1 = 1/n$$

Then eq.(3.33) becomes,

(3.35)
$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2 K(n+1)}{\rho_0 n}\rho_0^{1/n}\frac{d\rho_0}{dr}\right) = -4\pi G\,\rho_0$$

Now let,

(3.36)
$$\rho_0 = \rho_c \theta^n$$

Then eq(3.35) becomes,

(3.37)
$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2 K(n+1)}{n}(\rho_c \theta^n)^{(1-n)/n}\frac{d}{dr}\rho_c \theta^n\right) = -4\pi\,\rho_c \theta^n G$$

Now, pulling out all the constants, and noting that,

(3.38)
$$\frac{d}{dr}\theta^n = n\,\theta^{n-1}\frac{d\theta}{dr}$$

Eq.(3.37) becomes,

(3.39)
$$\left[\frac{n+1}{4\pi G}K\,\rho_c^{(\frac{1}{n}-1)}\right]\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\theta}{dr}\right) = -\theta^n$$

Finally, let,

(3.40) $$a = [\frac{n+1}{4\pi G} K \rho_c^{(1/n)-1}]^{1/2}$$

and,

(3.41) $$r = a\xi$$

Eq.(3.39) becomes,

(3.42) $$\frac{1}{\xi^2}\frac{d}{d\xi}(\xi^2 \frac{d\theta}{d\xi}) = -\theta^n$$

This is known as the Lane-Emden equation of index n. A more extensive discussion of these functions is given in appendix B. For now it is enough to state that for n<5 the equations have a zero at a finite value xi=xi_1. Since from eq.(3.36), theta is related to the density (rho_0), the first zero theta is a zero density, and therefore represents the surface of the star. The particular values needed are from eqs.(3.23b) and (3.24b),

(3.43) $$\gamma = \frac{5}{3}, n = 3/2, \xi_1 = 3.65375, \xi_1^2 |\theta'(\xi_1)| = 2.71406$$

and,

(3.44) $$\gamma = \frac{4}{3}, n = 3, \xi_1 = 6.89685, \xi_1^2 |\theta'(\xi_1)| = 2.01824$$

The last values in these sets will be used later in eq.(3.50). With these values it is possible to write values for the mass, M, and radius, R, of the star in terms of the central density, rho_c, and then to write the relationship between mass and radius. From eq.(3.41),

$$R = a\xi_1$$

(3.45)

and from eq.(3.28),

$$M = \int_0^R 4\pi r^2 \rho_0 dr$$

(3.46)

Now using eq.(3.36),

$$M = \int_0^R 4\pi r^2 \rho_c \theta^n dr$$

(3.47)

and using eq.(3.41),

$$M = 4\pi \rho_c a^3 \int_0^{\xi_1} \xi^2 \theta^n d\xi$$

(3.48)

Finally using eq.(3.42), gives,

$$M = -4\pi a^3 \rho_c \int_0^{\xi_1} \frac{d}{d\xi}(\xi^2 \frac{d\theta}{d\xi})d\xi$$

(3.49)

or,

$$M = 4\pi a^3 \rho_c \xi_1^2 |\theta'(\xi_1)|$$

(3.50)

Now from eq.(3.40), in the non-relativistic case, where gamma = 5/3 and n = 3/2 it can be seen that,

$$a = \rho_c^{-1/6} * constants$$

(3.51)

Then from eq.(3.45),

(3.52) $$R = \rho_c^{-1/6} * constants$$

and, from eq.(3.50) and eq.(3.51),

$$M = \rho_c^{1/2} * constants$$

(3.53)

and finally, using eq.(3.52) and eq.(3.53),

$$M = R^{-3} * constants$$
(3.54)

This is a very important result. It says that the mass of the star is inversely proportional to the volume of the star. So as mass is added to the star it will shrink. As it does so the space available to the electrons will decrease, and because of the uncertainty principle their momentum must increase. Eventually, they will approach the speed of light, and the ultra-relativistic approximation is appropriate. From eq.(3.40), in the ultra-relativistic case, where gamma = 4/3, and n = 3,

$$a = \rho_c^{-1/3} * constants$$
(3.55)

And with eq(3.50),

(3.56) M= constants

This result says that as the electrons become more relativistic, the radius becomes more sensitive to changes in mass. When a critical mass is reached, the radius will become zero, unless other forces stop the collapse . This is the Chandrasekhar limit. Specifically, starting with eq.(3.50) and using eq.(3.44),

(3.57) $$M_{Ch} = 4\pi\, a^3\, \rho_c * 2.01824$$

And using eq.(3.40) gives,

(3.58) $$M_{Ch} = 4\pi * 2.01824 * [\frac{K}{\pi G}]^{3/2}$$

Finally, using eq.(3.24a), with Y_e = 1/2, which is a good estimate for matter evolved to helium, and beyond, and using the values,

$m_B = 1.66057*10^{-24}$ g
$c = 3*10^{10}$ cm
$G = 6.670*10^{-8}$ dynes*cm^2/g^2
$h = 6.62*10^{-27}$ erg*sec
M_{Solar}= One solar mass = $1.989*10^{33}$ g

The result is,

(3.59) $$M_{Ch} = 1.46\ M_{Solar}$$

This is the Chandrasekhar limit.

The equation of state represented by eqs.(3.22), (3.23), and (3.24), is the equation that was used by Chandrasekhar in his pioneering analysis. Although it correctly predicts the collapse of the star, other refinements made since then have changed the exact value of the limiting mass, and the description of the star at lower masses. The two most important of these are the electrostatic correction, and the correction for inverse beta decay. These are discussed in Section V. Some of the other approximations that were made in the derivation, are explained as follows: In arriving at eq.(3.15), T is set to 0. This is justified for a number of reasons. First, this is the expected end state of the star, so this is just considering the simplest case first. Second, at very high densities the thermal pressure is small compared to that of the electron degeneracy pressure. An order of magnitude argument is as follows: The thermal energy is given as,

$$E_T = kT \approx 1.67 * 10^{-20} \, ergs \qquad (3.60)$$

where k, again, is the Boltzmann constant, k = $1.67*10^{-24}$ g/cm^3, and T has been estimated at T = 10,000 K. At low energies, the energy due to degeneracy can be given as,

$$E_e = \frac{p_F^2}{2 m_e} \qquad (3.61)$$

Where m_e is the electron mass, $m_e = 9.11*10^{-28}$ g. Now since P_F can be estimated using the uncertainty principle as,

$$p_F = h \, n_e^{1/3} \qquad (3.62)$$

However, a more precise value is obtained from eq.(3.15), where

$$p_F = h n_e^{1/3} \left(\frac{3}{8\pi}\right)^{1/3} \qquad (3.63)$$

Using Eq.(3.63), Eq.(3.61) becomes,

$$E_e = \frac{h^2 n_e^{2/3} (\frac{3}{8\pi})^{2/3}}{2 m_e} = 4.16 * 10^{-7} \, ergs \qquad (3.64)$$

Where n_e has been estimated as $5.99*10^{29}$ cm^{-3}, which corresponds to a density of 10^6 g/cm^3. So by comparing eq.(3.64) to eq.(3.60), the degeneracy energy is about 10^{13} times larger than the thermal energy.

Another approximation is made in arriving at eq.(3.17). It is assumed that all of the mass of the star is in the form of baryons, i.e. protons and neutrons. This is justified because the baryon mass, m_B, is about 1800 times larger than the electron mass, m_e. This also justifies the assumption that the degeneracy pressure is due entirely to the electrons. From eq.(3.64), the baryon energy would be,

$$E_B = \frac{h^2 n_B^{2/3} (\frac{3}{8\pi})^{2/3}}{2 m_B} = 3.67 * 10^{-10} \, ergs \qquad (3.65)$$

or about 1100 times smaller than the electron energy. Here n_B is the baryon number density, estimated at approximately twice the electron number density. At very high energies eq.(3.61) is no longer true, and the baryons do begin to provide a larger fraction of the pressure. This is discussed more in Section V. Finally, the star considered was non-rotating and had no magnetic field. This, of course, was done to limit the problem to the simplest case.

Section IV -A Simple Argument for the Existence of a Limit

This section gives a simple argument for the existence of the Chandrasekhar Limit.

Using eqs. (3.63) and (3.3) the Fermi energy can be written as,

(4.1) $$E_F = hcn_e^{1/3}\left(\frac{3}{8\pi}\right)^{1/3}$$

Now, taking the electron density as,

(4.2) $$n_e \approx N_e / R^3$$

where N_e is the total number of electrons in the star,

(4.3) $$E_F \approx \frac{hc\,N_e^{1/3}}{R}\left(\frac{3}{8\pi}\right)^{1/3}$$

The gravitational energy can be written as,

(4.4) $$E_G = -\frac{GMm_B}{R}$$

or as,

(4.5) $$E_G \approx -\frac{GN_B m_B^2}{R}$$

where N_B is the total number of baryons in the star. Now using,

(4.6)
$$N_B \approx 2 N_e$$

the total energy can be written as,

(4.7)
$$E_T = E_F + E_G \approx \frac{N_e^{1/3}(hc)(\frac{3}{8\pi})^{1/3} - N_e(2Gm_B^2)}{R}$$

This equation can be studied to see the nature of the equilibrium, which will occur at a minimum value of E_T. If N_e is small, so that E_T is positive, then E_T can be decreased by increasing R. This will happen until the electrons become non-relativistic and eq.(4.7), becomes invalid. However if N_e is large enough that E_T is negative, then the energy can only be decreased by decreasing R, and the star will contract to a point, if it is not stopped by other for

Section V - Improvements to the Basic Model

The most important improvement to Chandrasekhar's original work is the electrostatic correction. The major part of this correction arises because the protons are not evenly distributed throughout the star, but instead are grouped together into nuclei. A short discussion of this effect is given in the first part of this section. The second part will discuss inverse beta decay.

The average attractive force between a nucleus and an electron is,

(5.1) $$F_{B,e} = \frac{-Ze^2}{r_{B,e}^2}$$

where e is the electron charge, Ze is the average nuclear charge, and $r_{B,e}$ is the average electron-nucleus distance. The average repulsive force between two electron is given as,

(5.2) $$F_{e,e} = \frac{e^2}{r_{e,e}^2}$$

where $r_{e,e}$ is the average electron-electron distance. Now, because of the decreased density of positively charged particles,

(5.3) $$r_{B,e} = r_{e,e} Z^{1/3}$$

Combining eqs.(5.1), (5.2), and (5.3) gives,

(5.4) $$F_{B,e} = -Z^{1/3} F_{e,e}$$

Then the total electrostatic force is,

(5.5) $$F_{B,e} + F_{e,e} = (1 - Z^{1/3})\,F_{e,e}$$

So, the net force is attractive, for Z > 1. This has the effect of lowering the pressure at a given density. However, this is only a small fraction of the degeneracy pressure, which can be shown as follows,

The Fermi energy is given by eq.(3.65), and the electrostatic energy is given as,

(5.6) $$\frac{Ze^2}{r_{e,e}}$$

Now using,

(5.7) $$n_e^{1/3} = \frac{4\pi}{3} r_{e,e}$$

The result for the ratio of the Coulomb energy to the Fermi energy is,

(5.8) $$\frac{E_C}{E_F} = \frac{2\,Ze^2\,m_e}{h^2\,n_e^{1/3}}\left(\frac{3}{\pi}\right)^{1/3}$$

Now using e = 4.80*10^{-10} ($g^{1/2}cm^3/2s^{-1}$), m_e = 9.11*10^{-28} g, and using a density of 10^6 g /cm^3,
and Z =6 for carbon, gives,

$$\frac{E_C}{E_F} = \frac{Z}{n_e^{1/3}} * 9.7 * 10^6 = 6.9 * 10^{-3}$$

(5.9)

So, the Coulomb energy is less than 1% of the Fermi energy. Although the correction to the pressure is small, it is significant, and results in a smaller radius, and a larger central density for a given mass. This is because the radius is very sensitive to changes in pressure, at high densities.

Chandrasekhar's original model, with this small electrostatic correction, is still the standard model for white dwarf matter with a density between 10^4 g/cm^3 and 10^7 g/cm^3. Below 10^4 g/cm^3, the electrons can not be treated as a uniform gas, since shell effects become important. A statistical approximation for this low density case was carried out by Feynman, Metropolis, and Teller in 1949[9].

At densities higher than 10^7 g/cm^3, inverse beta decay must be considered. The reaction is,

(5.10) $$e^- + p^+ \rightarrow n + \gamma$$

Or, an electron and a proton yields a neutron and a neutrino. The minimum density for the onset of this reaction is found by setting the total energy of the electron and proton, equal to the total energy of the neutron. Since the baryons are non-relativistic at this energy, their kinetic energies are not included. Thus, where m_n is the neutron mass, and m_p is the proton mass.

(5.11) $$(m_e^2 c^4 + (p_F^2 c)^2)^{1/2} + m_p c^2 = m_n c^2$$

or:

(5.12) $$(m_e^2 + \frac{p_F^2}{c^2})^{1/2} = m_n - m_p$$

Now, from eq.(3.15),

(5.13) $$p_F^2 = (\frac{3 h^3}{8\pi})^{2/3} n_e^{2/3}$$

Substituting eq.(5.13) into eq.(5.12), and solving for n_e gives,

(5.14) $$n_e = ((m_n - m_p)^2 - m_e)^{3/2} \frac{8\pi c^3}{3h^3} = 7.4 * 10^{30}\, cm^{-3}$$

where, $m_p = 1.673*10^{-24}$ g = 1.007593 a.m.u., and m_n = 1.008982 a.m.u.. Now the density is given by,

(5.15) $$n_e m_p = 1.2 * 10^7\, g/cm^3$$

If the star were composed of free neutrons and protons, a phase change would take place at this density. Most of the electrons would be absorbed, greatly reducing the pressure, and the star would collapse until pressure from degenerate neutrons halted the collapse. In reality, the situation is not that simple. The star contains baryons in discrete nuclei. A number of models have emerged to deal with this.

In 1958, Harrison and Wheeler assumed that the star started as pure ^{56}Fe, which would be the lowest energy state at low densities.[10] They also used a continuous function for the mass of the nuclei. The result was that the equilibrium shifts towards heavier, more neutron rich nuclei, at higher densities.

In 1961, Hamada and Salpeter[5] improved on the model, by using discrete values for the nuclear masses.[11] The result was a number of phase changes, from one nucleus to the next, as the star becomes more dense. Starting with the reaction ^{56}Fe to ^{62}Ni, at about $8.1*10^6$ $g/cm^{3,}$ and ending with ^{118}Kr at about $4.4*10^{11}$ g/cm^3.

In 1971, Baym, Pethick and Sutherland, improved the formula slightly.[12] Their improvement involved using some better approximations.

These models all assume that the star can reach its lowest energy state. This may not be the case. Current stellar models suggest that stars that finish their lives as white dwarves, may have mostly carbon cores. And the reaction ^{12}C to ^{12}B to ^{12}Be does not take place until densities reach about $3.9*10^{10}$ g/cm^3.

Another factor to consider, however, is that the star will under-go some Pycnonuclear reactions. These are reactions in which the pressure is used to overcome the coulomb repulsion of the nuclei. These reactions would continue to evolve the star towards heavier elements. However, current calculations suggest that the time for these reactions to take place is too long to have a major impact. So, most astrophysicists feel that the stars are likely to be mostly carbon. A small minority feel that the stars may be composed of heavier elements, like iron.

All of the models predict some loss of pressure from inverse beta decay, which results in smaller radii, and larger central densities for a given mass.

At about $4*10^{11}$ g/cm^3, the models predict what is known as "neutron drip". This is the release of free neutrons from the nuclei. At these densities it becomes the lower energy state for the nuclei, instead of electron capture. From about $4*10^{11}$ g/cm^3 to about $4*10^{12}$ g/cm^3, the star is best described as nuclei, electrons and free neutrons in equilibrium. At higher densities, most of the pressure to support the star comes from degenerate neutrons, and the object is no longer described as a white dwarf, but instead, as a neutron star. The best models here treat the star as a degenerate neutron gas, in equilibrium with a small amount of electrons and protons.

Finally, when general relativity is considered, instability sets in above some threshold density, leading to the collapse of a white dwarf to form a neutron star. This is dependent, however, on the composition of the star. For heavier nuclei, the instability will never take place. But for carbon stars, the instability occurs at densities below neutron drip, specifically, at about $2.65*10^{10}$ g/cm^3. This may lead to the collapse of the star at these densities.

In short, although the description of the star at high densities becomes very complicated, Chandrasekhar's result of a limiting mass for white dwarf stars, still remains intact.

Appendix A - Polytropic Equation of State

Eq.(3.22) is a polytropic equation of state. A polytropic process, is any process, in which the heat capacity remains constant, that is,

(A.1) $$\frac{dQ}{dT} = C = constant$$

where Q is the heat content. It should be noted, that if C = 0, this is the special case of an adiabatic process, and if C is infinite, this is an isotropic process. In this section it will be shown that eq.(A.1) implies eq.(3.22).

First, a few basic definitions, and thermodynamic relations are needed. The first law of thermodynamics can be written as,

(A.2) $$dU = dQ - PdV$$

where, U is the internal energy. It can be given by,

(A.3) $$U = \int_0^T C_V \, dTordU = C_V \, dT$$

C_V, the heat capacity at constant volume, is defined as,

(A.4) $$C_V = \left(\frac{\partial Q}{\partial T}\right)_V$$

or, using eq.(A.2), and the fact that for an ideal gas C_V depends only on T,

(A.5) $$C_V = \left(\frac{\partial U}{\partial T}\right)_V = \left(\frac{\partial U}{\partial T}\right)_P$$

Also, the heat capacity at constant pressure is defined as,

(A.6)

$$C_P = \left(\frac{\partial Q}{\partial T}\right)_P$$

Finally, the ideal gas law is,

(A.7)

$$P = \frac{N}{V} kT$$

where, N is the total number of particles, and again n is the particle density, and V, the volume, related by,

(A.8)

$$N = nV$$

Now, starting with eq.(A.2), and taking the partial derivative of both sides with respect to T, at constant pressure, gives,

(A.9)

$$\left(\frac{\partial Q}{\partial T}\right)_P = \left(\frac{\partial U}{\partial T}\right)_P + P\left(\frac{\partial V}{\partial T}\right)_P$$

Using the definitions in eqs.(A.5) and (A.6), gives,

(A.10)

$$C_P = C_V + P\left(\frac{\partial V}{\partial T}\right)_P$$

and, using eq.(A.7), gives,

(A.11)

$$C_P = C_V + Nk$$

Now, returning to the first law, eq.(A.2), and using the definition in eq.(A.1), gives, (A.12) $$CdT - dU = PdV$$

or, with eq.(A.3),

(A.13) $$CdT - C_V\,dT = PdV$$

Now substituting eq.(A.7) into eq.(A.11) gives,

(A.14) $$P = (C_P - C_V)\frac{T}{V} \, or PdV = (C_P - C_V)T\frac{dV}{V}$$

Combining eq.(A.13) and eq.(A.14) gives,

(A.15) $$CdT - C_V\,dT = (C_P - C_V)T\frac{dV}{V}$$

Now, the polytropic gamma is defined as,

(A.16) $$\gamma = \frac{C_P - C}{C_V - C}$$

and solving for C_P,

(A.17) $$C_P = \gamma(C_V - C) + C$$

Here, again, if C is 0, then gamma is just the adiabatic gamma, and if C is infinite, then gamma is 1, as is the case in an isotropic process. Now, substituting eq.(A.17) into eq.(A.15) gives, (A.18) $$(C - C_V)dT = (\gamma(C_V - C) + C - C_V)T\frac{dV}{V}$$

or,

(A.19) $$\frac{dT}{T} + (\gamma - 1)\frac{dV}{V} = 0$$

Integrating gives,

(A.20) $$\ln(T) = (1 - \gamma)\ln(V) + constant$$

or,

(A.21) $$\ln(TV^{\gamma-1}) = constant$$

or,

(A.22) $$TV^{\gamma-1} = constant$$

Then, using eq.(A.7) gives,

(A.23) $$P = \frac{N}{V}k(constant * V^{1-\gamma})$$

or,

$$P = V^{-\gamma} * constant$$

(A.24)

or, using
eq.(A.8),

$$P = n^{\gamma} * constant$$

(A.25)

or, using eq.(3.17), where now $n_e = n$, and naming the constant K gives eq.(3.22).

$$P = K \rho_0^{\gamma}$$

(A.26)

Appendix B - The Lane-Emden Equation

The Lane-Emden equation, of index n, is given as,

(B.1)
$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n$$

It only has analytic solutions for a few indexes of n. These are 0, 1, and 5. For n = 0 the solution is,

(B.2)
$$\theta_0 = 1 - \frac{1}{6}\xi^2$$

For n = 1 the solution is,

(B.3)
$$\theta_1 = \frac{\sin\xi}{\xi}$$

And, for n = 5, the solution is,

(B.4)
$$\theta_5 = \frac{1}{\left(1+\frac{1}{3}\xi^2\right)^{1/2}}$$

For all other n, numerical methods must be used, and these results can be found in standard tables. For n greater than or equal to 5 there are no 0 values for theta.

In the following pages Mathematica has been used to generate a plot of the function for various values of n. The plots start at theta = .0001 to avoid infinities at 0, and the real part of the result is used for the plot, in order to remove infinitesimal imaginaries that the approximation introduces, and for n = 1/2, the graph was truncated, due to lack of memory. For n = 3/2, and n = 3, the indexes referenced in the main text, the standard table[13] values for the first zeros, and the expression,

$$\xi_1^2 |\theta'(\xi_1)| \tag{B.5}$$

were checked. Both the first zero, and the value of eq.(B.5) were found to be in agreement with the tabulated values.

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